4.2 Reduction of order

Consider a 2^{nd} order linear homogeneous ODE:

$$y'' + P(x)\,y' + q(x)\,y = 0 \quad (1)$$

By our theory, general solution:

$$y = c_1 y_1 + c_2 y_2 \quad (y_1, y_2 \text{ lin. ind.})$$

Suppose that y_1 is known, find y_2.

Idea: $y_2(x) = v(x)\,y_1(x)$

$$y_2' = v' y_1 + v y_1'$$

$$y_2'' = v'' y_1 + 2v' y_1' + v y_1''$$

Sub in (1) yields:

$$\underbrace{v'' y_1 + 2v' y_1' + v y_1''}_{y_2''} + \underbrace{P v' y_1 + P v y_1'}_{P(x) y_2'} + \underbrace{q v y_1}_{q(x) y_2} = 0$$

$$v(\underbrace{y_1'' + P y_1' + q y_1}_{0}) + v'' y_1 + 2v' y_1' + P v' y_1 = 0$$

$$v'' y_1 + 2v' y_1' + P v' y_1 = 0 \quad (*)$$

$\boxed{w = v'}$, $w' = v''$

$$(*) \iff w' y_1 + w(2y_1' + P y_1) = 0 \quad (**)$$

$(**)$ 1^{st} order (Reduction) linear eq.

$\Rightarrow$ Integrating factor

$$w' + w\left(\underbrace{2\frac{y_1'}{y_1} + P}\right) = 0 \quad (***)$$

Integrating factor : $\mu = e^{\int 2y_1'/y_1 + P}$
$$= y_1^2 \, e^{\int P(x)dx}$$

$$(***) \iff (w\mu)' = 0$$
$$\iff w\mu = c$$
$$\iff w = \frac{c}{\mu} = \frac{c}{y_1^2 \, e^{\int P(x)dx}}$$

$$\left.\begin{array}{l} \boxed{V(x) = \int w = \int \frac{1}{y_1^2 e^{\int P}}} \\ \boxed{y_2(x) = V(x)\, y_1(x)} \end{array}\right\}$$

<u>Ex</u> : $y'' - 4y' + 4y = 0$
$y_1 = e^{2x}$ is a solution, find y_2.

<u>Solution</u> : $P(x) = -4$
$$y_1^2 e^{\int P} = e^{4x} e^{-4x} = 1.$$

$$V(x) = \int \frac{1}{1} dx = x$$

$$y_2 = V(x)\, y_1(x) = x\, e^{2x}$$

<u>Ex</u> : $\underbrace{x y'' + y'}_{} = 0$, $y_1 = \ln(x)$, find y_2.
$x \neq 0$
$$y'' + \frac{1}{x} y' = 0, \quad P(x) = \frac{1}{x}, \quad \int \frac{1}{x} dx = \ln x$$

$$y_1^2 \, e^{\int P(x)dx} = (\ln(x))^2 e^{\ln x} = x(\ln x)^2$$

$$v(x) = \int \frac{1}{x(\ln x)^2}\,dx \quad , \quad u = \ln x,\ du = \frac{1}{x}\,dx$$

$$= \int \frac{1}{u^2}\,du = -\frac{1}{u} = -\frac{1}{\ln x}$$

$$y_2 = v(x)\,y_1(x) = -\frac{1}{\ln(x)} \cdot \ln(x) = -1.$$

4.3 Homogeneous Linear Constant Coefficient Equations

⊕ Second order case:

$$a y'' + b y' + c y = 0 \quad (1)$$

a, b, c are constants

- Solutions include factor e^{rx}
- Consider the characteristic polynomial

$$a r^2 + b r + c = 0 \quad (2)$$

Explanation: $y = e^{rx}$, $y' = r e^{rx}$, $y'' = r^2 e^{rx}$

Sub in (1) : $a r^2 e^{rx} + b r e^{rx} + c e^{rx} = 0$

$$\Leftrightarrow e^{rx}(a r^2 + b r + c) = 0$$

$$\Leftrightarrow a r^2 + b r + c = 0$$

- Quadratic polynomial :

Discriminant : $\Delta = b^2 - 4ac$; $r = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

Case 1 : $\Delta > 0$, 2 real roots $r_1 \neq r_2$

$\Rightarrow$ 2 ind. sol $e^{r_1 x}$, $e^{r_2 x}$

$\Rightarrow$ General sol: $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Case 2: $\Delta = 0$, Real double root $r_1 = r_2$

$\Rightarrow$ 1 ind. sol $e^{r_1 x}$

$\Rightarrow$ (by reduction of order) 2nd sol: $xe^{r_1 x}$

$\Rightarrow$ General sol: $y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$.

Case 3: $\Delta < 0$, no real roots, but complex roots

$r = \alpha \pm i\beta$

$\Rightarrow$ 2 ind. sol. $\underline{e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x)}$

$\Rightarrow$ General sol: $y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$

Explanation: Euler formula:

$$\left.\begin{aligned} e^{i\beta} &= \cos\beta + i\sin\beta \\ e^{-i\beta} &= \cos\beta - i\sin\beta \end{aligned}\right\} \Rightarrow e^{(\alpha \pm i\beta)x} \ldots$$

Ex: $r^2 + 1 = 0$, $r = \pm i$, $i = \sqrt{-1}$

Ex: $y'' + y = 0$,

Characteristic polynomial:

$r^2 + 1 = 0$, $r = \pm i$, $\alpha = 0$, $\beta = 1$.

$y_1 = e^{0x}\sin(1x) = \sin x$

$y_2 = e^{0x}\cos(1x) = \cos x$.

Ex: $y'' - y' - 6y = 0$

Char. poly: $r^2 - r - 6 = 0 = (r+2)(r-3)$

Roots $= -2, 3.$

$y_1 = e^{-2x}, \quad y_2 = e^{3x}$

$y = c_1 e^{-2x} + c_2 e^{3x}.$

⊕ The higher order case:

$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$

$a_n, \ldots a_1, a_0$ are all constants.

- Char. poly: $a_n r^n + \cdots + a_1 r + a_0 = 0 = P(r)$
- Factorize $P(r)$ into linear and quad terms

$P(r) = a_n (r-r_1)^{m_1} \cdots (r-r_k)^{m_k} (r^2 - 2\alpha_1 r + \alpha_1^2 + \beta_1^2)^{n_1}$
$\cdots (r^2 - 2\alpha_\ell r + \alpha_\ell^2 + \beta_\ell^2)^{n_\ell}$

Each factor contributes some solutions

The general sol is just the linear combination of them all.

(*) $(r-r_1)^{m_1} \Rightarrow e^{r_1 x}, x e^{r_1 x}, \ldots, x^{m_1 - 1} e^{r_1 x}$

(**) $(r^2 - 2\alpha r + \alpha^2 + \beta^2)^n$

$\Rightarrow y = (c_1 + c_2 x + \cdots + c_n x^{n-1}) e^{\alpha x} \cos(\beta x)$
$+ (d_1 + d_2 x + \cdots + d_n x^{n-1}) e^{\alpha x} \sin(\beta x)$

Ex: $y''' - 4y'' - 5y' = 0$

Char poly: $r^3 - 4r^2 - 5r = 0 = r(r^2 - 4r - 5) = 0$
$= r(r-5)(r+1) = 0$

Roots: $r = 0, 5, -1$.

General sol: $y = c_1 e^{0x} + c_2 e^{5x} + c_3 e^{-x}$

$= c_1 + c_2 e^{5x} + c_3 e^{-x}$.

<u>Ex</u>: $y''' - 5y'' + 3y' + 9y = 0$.

Char poly: $r^3 - 5r^2 + 3r + 9 = 0$

How to find a linear factor?

<u>Ans</u>: Rational root test

$P(r) = a_n r^n + \cdots a_1 r + a_0$

$a_n, \ldots a_1, a_0$ are integers

If $r = \frac{p}{q}$ is a rational root in lowest term

then p divides evenly into a_0 and

q divides evenly into a_n.

In this example: $a_0 = 9$, $a_n = 1 = a_3$

$\Rightarrow q = 1$

$p \mid 9 \Rightarrow p = \pm 1, \pm 3, \pm 9$

You can check that $\frac{p}{q} = -1$ is a root.

$r^3 - 5r^2 + 3r + 9 = r^3 + r^2 - 6r^2 - 6r + 9r + 9$

$= (r+1)(r^2 - 6r + 9) = (r+1)(r-3)^2$

$y = c_1 e^{-x} + c_2 e^{3x} + c_3 x e^{3x}$.